On the asymptotic behavior of the quasi-static problem for a linear viscoelastic fluid

Mauro Fabrizio[a], Barbara Lazzari[a,*], Roberta Nibbi[a]

[a]*Department of Mathematics, University of Bologna, 5 Piazza di Porta S. Donato, 40126 Bologna, Italy*

Abstract

In this paper we study the quasi-static problem for a viscoelastic fluid by means of the concept of minimal state. This implies the use of a different free energy defined in a wider space of data. The existence and uniqueness is proved in this new space and the asymptotic decay for the problem with non vanishing supplies is obtained for a large class of memory kernels, including those presenting an exponential or polynomial decay.

Keywords: Asymptotic decay, Viscoelastic fluids, Quasi-static problem
2000 MSC: 74D05, 76A10, 30E20, 35B40.

1. Introduction

When studying materials with memory, the classical approach is based on the histories of the deformation gradient. In Del Piero and Deseri (1997) it has been shown that different histories may lead to the same response of the material[1] and a new concept of state, relying on the minimal information required to determine the further behavior of the material, has been introduced for linear viscoelastic models. Furthermore, it is well known that several free energies can be defined for materials with memory. The family of the free energies is a convex set which has a minimum and a maximum element ψ_{min}, ψ_{max}. It follows that, for any free energy ψ, the state domain $\mathcal{H}_\psi$, for which ψ is finite, is such that $\mathcal{H}_{\psi_{max}} \subset \mathcal{H}_\psi \subset \mathcal{H}_{\psi_{min}}$.

In this paper, making use of the concept of minimal state, we study the quasi-static problem for a linear incompressible viscoelastic fluid and prove that it admits a unique solution belonging to $\mathcal{H}_{\psi_{min}}$ if the memory kernel satisfies only the restrictions imposed by the Laws of Thermodynamics and the data belong to the dual space of $\mathcal{H}_{\psi_{min}}$, which is the widest space that one can expect.

As for the long time behavior, many results have been established for the dynamic problem with memory kernels exhibiting exponential or polynomial decay, but in general for vanishing supplies.

Recently Messaoudi (2008) has proposed a unified approach and proved that, in the evolutive problem with vanishing past histories and no external forces, the energy has the same type of temporal decay of the memory kernel, which is not necessarily an exponential or polynomial decay.

Here, we restrict ourselves to the quasi-static approximation of the problem and obtain a temporal decay similar to the one obtained by Messaoudi (2008), but in presence of supplies and in a wider space of initial past histories.

This is a first step in order to apply both the concept of minimal state and the unified approach to more general problems in viscoelasticity.

The paper is organized as follows. In Section 2 we recall some properties of linear viscoelastic fluids and introduce the concept of minimal state. In Section 3 we consider the quasi-static problem and establish its well posedness, while in Section 4 we present our results on the asymptotic behavior.

[1]For completeness, we recall that a first contribution in this direction was presented in Banfi (1977).

2. Basic assumptions for linear viscoelastic fluids

In this work we consider a viscoelastic fluid defined by means of the constitutive equation for the Cauchy stress tensor

$$\mathbf{T}(\mathbf{x}, t) = -p(\mathbf{x}, t)\mathbf{I} + \mathbf{T}_E(\mathbf{E}_r^t(\mathbf{x})),$$

where p represents the *pressure*, $\mathbf{I}$ denotes the unit second order tensor and the *extra stress* $\mathbf{T}_E$ is a function of the *relative strain history* $\mathbf{E}_r^t(\mathbf{x}, s) = \mathbf{E}(\mathbf{x}, t - s) - \mathbf{E}(\mathbf{x}, t)$ at any fixed point of the material (Joseph (1990)). These fluids are described by the classical Boltzmann-Volterra constitutive equation between the current value of the extra stress $\mathbf{T}_E(\mathbf{x}, t)$ and the relative strain history $\mathbf{E}_r^t(\mathbf{x}, s)$

$$\mathbf{T}_E(t) = 2 \int_0^{+\infty} \mu'(\mathbf{x}, s)\mathbf{E}_r^t(\mathbf{x}, s)\, ds. \tag{2.1}$$

Here μ' is a constitutive function, called *memory kernel*, such that the *shear relaxation function*

$$\mu(\mathbf{x}, s) = -\int_s^{+\infty} \mu'(\mathbf{x}, \xi)d\xi \qquad s \geq 0$$

belongs to $L^1(\mathbb{R}^+; L^\infty(\Omega))$. As proved in Fabrizio and Lazzari (1993), the thermodynamic principles provide, almost everywhere in Ω, the following restriction on its Fourier cosine transform:

$$\mu_c(\mathbf{x}, \omega) > 0 \qquad \forall \omega \in \mathbb{R}^+. \tag{2.2}$$

For these materials the physical state at time t is identified through the *mass density* ρ and the relative strain history $\mathbf{E}_r^t(\mathbf{x}, \cdot)$ at the time t.

In the following, we consider incompressible fluids, for which we have $\nabla \cdot \mathbf{v} = 0$, where $\mathbf{v}$ denotes the *velocity*; therefore, the state is defined only by means of the relative strain history.

By introducing the vector space of the admissible relative strain histories

$$\Gamma_r = \left\{ \mathbf{E}_r^t : \Omega \times \mathbb{R}^{++} \to Sym; \left| \int_0^{+\infty} \int_\Omega \mu'(\mathbf{x}, \xi + \tau)\mathbf{E}_r^t(\mathbf{x}, \xi)d\mathbf{x}d\xi \right| < +\infty, \quad \forall \tau \geq 0 \right\},$$

it is possible to give the following equivalence relation (see Deseri et al. (2006)).

Definition 2.1. Let $\mathbf{x} \in \Omega$. Two relative strain histories $\mathbf{E}_{r_j}^t(\mathbf{x}, \cdot)$, $(j = 1, 2)$ are said to be equivalent if and only if

$$\int_0^{+\infty} \mu'(\mathbf{x}, \xi + \tau) \left[\mathbf{E}_{r_1}^t(\mathbf{x}, \xi) - \mathbf{E}_{r_2}^t(\mathbf{x}, \xi) \right] d\xi = \mathbf{0}, \qquad \forall \tau > 0.$$

Let us introduce

$$\breve{\mathbf{I}}^t(\mathbf{x}, \tau) = -2 \int_0^{+\infty} \mu'(\mathbf{x}, \xi + \tau)\mathbf{E}_r^{\mathbf{x}, t}(\xi)d\xi. \tag{2.3}$$

As a consequence of Definition 2.1, $\breve{\mathbf{I}}^t$ characterizes the equivalence class in the space of the admissible relative strain histories and hence we will call it the *minimal state*.

3. Application to the quasi-static problem

In this section we study the quasi-static problem for incompressible linear viscoelastic fluids using the minimal state $\breve{\mathbf{I}}^t$. In fact, thanks to (2.1) and (2.3), the extra stress can be written in the following manner

$$\mathbf{T}_E(\mathbf{x}, t) = -\breve{\mathbf{I}}^t(\mathbf{x}, 0) = 2 \int_0^t \mu(\mathbf{x}, s)\dot{\mathbf{E}}^t(\mathbf{x}, s)ds - \breve{\mathbf{I}}^0(\mathbf{x}, t)$$

where $\breve{\mathbf{I}}^0$ is related to the initial past history and therefore is a known function.

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Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. The linear approximation of the quasi-static boundary value problem with Dirichlet conditions is

$$0 = -\nabla p(\mathbf{x}, t) + \nabla \cdot \int_0^t \mu(\mathbf{x}, s)\nabla \mathbf{v}^t(\mathbf{x}, s)ds - \nabla \cdot \breve{\mathbf{I}}^0(\mathbf{x}, t) + \mathbf{f}(\mathbf{x}, t) \tag{3.1}$$

$$\nabla \cdot \mathbf{v}(\mathbf{x}, t) = 0, \quad \mathbf{v}(\mathbf{x}, t)_{|\partial\Omega} = \mathbf{0}. \tag{3.2}$$

In order to give a precise formulation of problem (3.1) – (3.2) we introduce the space

$$J(\Omega) = \{\mathbf{v} \in C_0^\infty(\Omega); \ \nabla \cdot \mathbf{v} = 0\}$$

and denote by $\overset{\circ}{L}{}^2(\Omega)$ and $\overset{\circ}{H}{}_0^1(\Omega)$ the closure of $J(\Omega)$ in the L^2 and H^1 norms respectively. Moreover we consider the spaces

$$\mathcal{H}_\mu(\mathbb{R}^+, \Omega) = \left\{ \mathbf{v} \in L_{loc}^2(\mathbb{R}^+; \overset{\circ}{H}{}_0^1(\Omega)); \int_0^{+\infty}\int_0^{+\infty}\int_\Omega \mu(\mathbf{x}, |\tau - \tau'|)\nabla\mathbf{v}(\mathbf{x}, \tau) \cdot \nabla\mathbf{v}(\mathbf{x}, \tau')d\mathbf{x}d\tau d\tau' < +\infty \right\}, \tag{3.3}$$

$$\mathcal{S}_\mu(\mathbb{R}^+, \Omega) = \left\{ \breve{\mathbf{I}}^0 \in L_{loc}^2(\mathbb{R}^+; L^2(\Omega)); \int_0^{+\infty}\int_0^{+\infty}\int_\Omega \tilde\mu(\mathbf{x}, |\tau - \tau'|)\breve{\mathbf{I}}^0(\mathbf{x}, \tau) \cdot \breve{\mathbf{I}}^0(\mathbf{x}, \tau')d\mathbf{x}d\tau d\tau' < +\infty \right\}, \tag{3.4}$$

where $\tilde\mu$ is defined by

$$\frac{1}{4}\int_{-\infty}^{+\infty} \mu(\mathbf{x}, |\tau - \tau'|)\tilde\mu(\mathbf{x}, |\tau'|)d\tau' = \delta(\tau).$$

Remark 3.1. If the kernel $\mu \in L^\infty(\Omega, L^1(\mathbb{R}^+))$ satisfies the thermodynamic condition (2.2) almost everywhere in Ω, then $\mathcal{H}_\mu$ and $\mathcal{S}_\mu$ are Hilbert spaces. In fact it is possible to define the spaces $\mathcal{H}_\mu$ and $\mathcal{S}_\mu$ in the frequency domain (see Deseri et al. (2006)) by observing that

$$\int_0^{+\infty}\int_0^{+\infty}\int_\Omega \mu(\mathbf{x}, |\tau - \tau'|)\nabla\mathbf{v}(\mathbf{x}, \tau) \cdot \nabla\mathbf{v}(\mathbf{x}, \tau')d\mathbf{x}d\tau d\tau' = \frac{1}{\pi}\int_{-\infty}^{+\infty}\int_\Omega \mu_c(\mathbf{x}, \omega)|\nabla\mathbf{v}_F(\mathbf{x}, \omega)|^2 d\mathbf{x}d\omega, \tag{3.5}$$

$$\int_0^{+\infty}\int_0^{+\infty}\int_\Omega \tilde\mu(\mathbf{x}, |\tau - \tau'|)\breve{\mathbf{I}}^0(\mathbf{x}, \tau) \cdot \breve{\mathbf{I}}^0(\mathbf{x}, \tau')d\mathbf{x}d\tau d\tau' = \frac{1}{\pi}\int_{-\infty}^{+\infty}\int_\Omega \frac{1}{\mu_c(\mathbf{x}, \omega)}|\breve{\mathbf{I}}_F^0(\mathbf{x}, \omega)|^2 d\mathbf{x}d\omega, \tag{3.6}$$

where the index $_F$ denotes the Fourier transform.

We finally recall that the left-hand side of (3.5) is the expression of the minimal free energy introduced in Breuer and Onat (1964a) and Breuer and Onat (1964b).

Definition 3.1. A function $\mathbf{v} \in \mathcal{H}_\mu(\mathbb{R}^+, \Omega)$ is said to be a weak solution in the sense of the Virtual Power Principle of the problem (3.1) – (3.2) if

$$2\int_0^{+\infty}\int_\Omega\int_0^t \mu(\mathbf{x}, t-s)\nabla\mathbf{v}(\mathbf{x}, s)ds \cdot \nabla\mathbf{w}(\mathbf{x}, t)d\mathbf{x}dt$$

$$= \int_0^{+\infty}\int_\Omega \breve{\mathbf{I}}^0(\mathbf{x}, t) \cdot \nabla\mathbf{w}(\mathbf{x}, t)d\mathbf{x}dt - \int_0^{+\infty}\int_\Omega \mathbf{f}(\mathbf{x}, t) \cdot \mathbf{w}(\mathbf{x}, t)d\mathbf{x}dt \tag{3.7}$$

for any $\mathbf{w} \in \mathcal{H}_\mu(\mathbb{R}^+, \Omega)$.

Remark 3.2. Given a vector $\mathbf{f}$, let $\nabla \times \mathbf{a}$ and $\nabla\phi$ be its solenoidal and irrotational components, respectively, in the Helmholtz decomposition, i.e. $\mathbf{f} = \nabla \times \mathbf{a} + \nabla\phi$. Since $\mathbf{w} \in \mathcal{H}_\mu(\mathbb{R}^+, \Omega)$, the last integral in (3.7) can be rewritten as

$$\int_0^{+\infty}\int_\Omega \mathbf{f}(\mathbf{x}, t) \cdot \mathbf{w}(\mathbf{x}, t)d\mathbf{x}dt = \int_0^{+\infty}\int_\Omega \nabla \times \mathbf{a}(\mathbf{x}, t) \cdot \mathbf{w}(\mathbf{x}, t)d\mathbf{x}dt.$$

Moreover, introducing the skew tensor $\mathbf{A}$, defined through the relation $\mathbf{A}\mathbf{w} = \mathbf{a} \times \mathbf{w}$, we have $\nabla \cdot \mathbf{A} = \nabla \times \mathbf{a}$ so that

$$\int_0^{+\infty} \int_\Omega \mathbf{f}(\mathbf{x}, t) \cdot \mathbf{w}(\mathbf{x}, t) d\mathbf{x} dt = -\int_0^{+\infty} \int_\Omega \mathbf{A}(\mathbf{x}, t) \cdot \nabla \mathbf{w}(\mathbf{x}, t) d\mathbf{x} dt.$$

By virtue of the previous remark, relation (3.7) becomes

$$\int_0^{+\infty} \int_\Omega \int_0^t \mu(\mathbf{x}, t-s) \nabla \mathbf{v}(\mathbf{x}, s) ds \cdot \nabla \mathbf{w}(\mathbf{x}, t) d\mathbf{x} dt = \int_0^{+\infty} \int_\Omega \breve{\mathbf{J}}^0(\mathbf{x}, t) \cdot \nabla \mathbf{w}(\mathbf{x}, t) d\mathbf{x} dt, \qquad (3.8)$$

where $\breve{\mathbf{J}}^0 = \breve{\mathbf{I}}^0 - \mathbf{A}$.

Theorem 3.1. *Problem* (3.1) – (3.2) *with* $\breve{\mathbf{J}}^0 \in \mathcal{S}_\mu(\mathbb{R}^+, \Omega)$ *admits a unique solution, according to Definition 3.1, if* $\mu \in L^1(\mathbb{R}^+, L^\infty(\Omega))$ *and* (2.2) *holds almost everywhere in* Ω.

Proof. Let us consider the Fourier transform of system (3.1) – (3.2)

$$\begin{aligned}
\nabla \cdot [\mu_F(\mathbf{x}, \omega) \nabla \mathbf{v}_F(\mathbf{x}, \omega)] - \nabla [p_F(\mathbf{x}, \omega) - \phi_F(\mathbf{x}, \omega)] &= \nabla \cdot \breve{\mathbf{J}}_F^0(\mathbf{x}, \omega) \\
\nabla \cdot \mathbf{v}_F(\mathbf{x}, \omega) = 0, \quad \mathbf{v}_F(\mathbf{x}, \omega) \mid_{\partial\Omega} &= \mathbf{0}
\end{aligned} \qquad (3.9)$$

and introduce the bilinear form

$$b_\omega(\mathbf{v}_F, \mathbf{w}_F) = \int_\Omega \mu_c(\mathbf{x}, \omega) \nabla \mathbf{v}_F(\mathbf{x}, \omega) \left[\nabla \mathbf{w}_F(\mathbf{x}, \omega) \right]^* d\mathbf{x}$$

for any fixed $\omega \in \mathbb{R}$, where the index * denotes the complex conjugate. The hypotheses on the kernel μ ensure that b_ω is bounded and coercive in $\overset{\circ}{H}_0^1(\Omega)$ since

$$k_1(\omega) \|\nabla \mathbf{v}_F(\omega)\|^2 \le b_\omega(\mathbf{v}_F, \mathbf{v}_F) \le k_2(\omega) \|\nabla \mathbf{v}_F(\omega)\|^2,$$

where $k_1(\omega)$ and $k_2(\omega)$ are the essential infimum and the essential supremum of $\mu_c(\mathbf{x}, \omega)$ on Ω respectively.

Therefore, thanks to the Lax-Milgram theorem, system (3.9) admits, for any fixed $\omega \in \mathbb{R}$, one and only one solution belonging to $\overset{\circ}{H}_0^1(\Omega)$ if the supply $\nabla \cdot \breve{\mathbf{J}}_F^0(\cdot, \omega)$ belongs to the dual space of $\overset{\circ}{H}_0^1(\Omega)$.

Moreover, if we rewrite (3.8), by virtue Plancherel's theorem, as follows

$$\int_{-\infty}^{+\infty} \int_\Omega \mu_F(\mathbf{x}, \omega) \nabla \mathbf{v}_F(\mathbf{x}, \omega) \cdot \left[\nabla \mathbf{w}_F(\mathbf{x}, \omega) \right]^* d\mathbf{x} d\omega = \int_{-\infty}^{+\infty} \int_\Omega \breve{\mathbf{J}}_F^0(\mathbf{x}, \omega) \cdot \left[\nabla \mathbf{w}_F(\mathbf{x}, \omega) \right]^* d\mathbf{x} d\omega \qquad (3.10)$$

and choose $\mathbf{w}_F = \mathbf{v}_F$ in (3.10), we obtain

$$\begin{aligned}
\int_{-\infty}^{+\infty} \int_\Omega \mu_c(\mathbf{x}, \omega) \mid \nabla \mathbf{v}_F(\mathbf{x}, \omega) \mid^2 d\mathbf{x} d\omega &= \int_{-\infty}^{+\infty} \int_\Omega \frac{\breve{\mathbf{J}}_F^0(\mathbf{x}, \omega)}{\sqrt{\mu_c(\mathbf{x}, \omega)}} \cdot \sqrt{\mu_c(\mathbf{x}, \omega)} \left[\nabla \mathbf{v}_F(\mathbf{x}, \omega) \right]^* d\mathbf{x} d\omega \\
&\le \left[\int_{-\infty}^{+\infty} \int_\Omega \frac{1}{\mu_c(\mathbf{x}, \omega)} \mid \breve{\mathbf{J}}_F^0(\mathbf{x}, \omega) \mid^2 d\mathbf{x} d\omega \right]^{1/2} \left[\int_{-\infty}^{+\infty} \int_\Omega \mu_c(\mathbf{x}, \omega) \mid \nabla \mathbf{v}_F(\mathbf{x}, \omega) \mid^2 d\mathbf{x} d\omega \right]^{1/2}.
\end{aligned}$$

Hence, thanks to (3.5) and (3.6), $\mathbf{v} \in \mathcal{H}_\mu(\mathbb{R}^+, \Omega)$ if $\breve{\mathbf{J}}^0 \in \mathcal{S}_\mu(\mathbb{R}^+, \Omega)$ because

$$\int_{-\infty}^{+\infty} \int_\Omega \mu_c(\mathbf{x}, \omega) \mid \nabla \mathbf{v}_F(\mathbf{x}, \omega) \mid^2 d\mathbf{x} d\omega \le \int_{-\infty}^{+\infty} \int_\Omega \frac{1}{\mu_c(\mathbf{x}, \omega)} \mid \breve{\mathbf{J}}_F^0(\mathbf{x}, \omega) \mid^2 d\mathbf{x} d\omega. \qquad (3.11)$$

Since $\breve{\mathbf{J}}^0 = \breve{\mathbf{I}}^0 - \mathbf{A}$, where $\nabla \cdot \mathbf{A}$ is the solenoidal part of the external force $\mathbf{f}$, we conclude that the virtual power solution of problem (3.1) – (3.2) belongs to $\mathcal{H}_\mu(\mathbb{R}^+; \Omega)$ if the initial datum $\breve{\mathbf{I}}^0 \in \mathcal{S}_\mu(\mathbb{R}^+; \Omega)$ and the solenoidal component $\mathbf{f}_s$ of $\mathbf{f}$ belongs to the space $L_{loc}^2(\mathbb{R}^+; H'(\Omega))$ where $H'(\Omega)$ is the dual of $\overset{\circ}{H}_0^1(\Omega)$ and there exists a skew tensor $\mathbf{A} \in \mathcal{S}_\mu(\mathbb{R}^+, \Omega)$ such that $\mathbf{f}_s = \nabla \cdot \mathbf{A}$. $\qquad \square$

4. Asymptotic behavior

Equation (3.1), in absence of external forces, can be rewritten in terms of the minimal state (2.3) as follows

$$\nabla \cdot \breve{\mathbf{I}}^t(\mathbf{x}, 0) + \nabla p(\mathbf{x}, t) = 0. \tag{4.1}$$

It is therefore necessary to assign the law governing the evolution in time of $\breve{\mathbf{I}}^t$ which, taking into account definition (2.3), is given by

$$\frac{\partial}{\partial t} \breve{\mathbf{I}}^t(\mathbf{x}, \tau) = \frac{\partial}{\partial \tau} \breve{\mathbf{I}}^t(\mathbf{x}, \tau) - 2\mu(\mathbf{x}, \tau)\nabla \mathbf{v}(\mathbf{x}, t), \qquad \breve{\mathbf{I}}^0(\mathbf{x}, \tau) = \mathbf{I}_0(\mathbf{x}, \tau), \tag{4.2}$$

where $\mathbf{I}_0(\mathbf{x}, \tau)$ is a known function on $\Omega \times \mathbb{R}^+$.

Theorem 3.1 ensures that problem (4.1), (4.2) and (3.2) admits a unique solution, according to Definition 4.1, whenever the initial datum $\mathbf{I}_0$ belongs to $\mathcal{S}_\mu(\mathbb{R}^+, \Omega)$. In this section we will study the connection between the asymptotic behavior of this solution and that of the memory kernel μ. To this aim we restrict ourselves to memory kernels satisfying almost everywhere in Ω the following restrictions

$$\mu'(\mathbf{x}, t) < 0, \quad \mu''(\mathbf{x}, t) \geq 0, \quad \mu''(\mathbf{x}, t)) \geq -\xi(t)\mu'(\mathbf{x}, t)), \qquad t \geq 0 \tag{4.3}$$

where ξ is a positive, non-increasing differential function.

Examples of such kernels can be found for example in Messaoudi (2008); in particular, a kernel presents an exponential or polynomial decay when ξ is a constant function or $\xi(t) = c(1 + t)^{-1}$, respectively.

Let us now consider problem (4.1), (4.2) and (3.2) with initial datum $\mathbf{I}_0$ belonging to the subspace[2] of $\mathcal{S}_\mu(\mathbb{R}^+, \Omega)$ defined by

$$\mathcal{F}_\mu(\mathbb{R}^+, \Omega) = \left\{ \breve{\mathbf{I}}^0 \in \mathcal{S}_\mu(\mathbb{R}^+, \Omega); \int_0^\infty \int_\Omega \frac{1}{-\mu'(\mathbf{x}, \tau)} \left| \frac{\partial}{\partial \tau} \breve{\mathbf{I}}^0(\mathbf{x}, \tau) \right|^2 d\mathbf{x} d\tau < \infty \right\} \tag{4.4}$$

and introduce the energy functional

$$\Psi(t) = \Psi(\breve{\mathbf{I}}^t) = -\frac{1}{4} \int_0^\infty \int_\Omega \frac{1}{\mu'(\mathbf{x}, \tau)} \left| \frac{\partial}{\partial \tau} \breve{\mathbf{I}}^t(\mathbf{x}, \tau) \right|^2 d\mathbf{x} d\tau. \tag{4.5}$$

If $\breve{\mathbf{I}}^t \in \mathcal{F}_\mu(\mathbb{R}^+, \Omega)$ this functional satisfies

$$\frac{d}{dt}\Psi(t) = -\int_\Omega \breve{\mathbf{I}}^t(\mathbf{x}, 0) \cdot \nabla \mathbf{v}(\mathbf{x}, t) d\mathbf{x} - \frac{1}{4} \int_0^\infty \int_\Omega \frac{\mu''(\mathbf{x}, \tau)}{[\mu'(\mathbf{x}, \tau)]^2} \left| \frac{\partial}{\partial \tau} \breve{\mathbf{I}}^t(\mathbf{x}, \tau) \right|^2 d\mathbf{x} d\tau + \frac{1}{4} \int_\Omega \frac{1}{\mu'(\mathbf{x}, 0)} \left| \frac{\partial}{\partial \tau} \breve{\mathbf{I}}^t(\mathbf{x}, 0) \right|^2 d\mathbf{x}; \tag{4.6}$$

moreover, fixed $T_0 > 0$, there exists $\alpha_{T_0} > 1$ such that

$$\Psi(t) \leq -\frac{\alpha_{T_0}}{4} \int_0^t \int_\Omega \frac{1}{\mu'(\mathbf{x}, \tau)} \left| \frac{\partial}{\partial \tau} \breve{\mathbf{I}}^t(\mathbf{x}, \tau) \right|^2 d\mathbf{x} d\tau \qquad \forall t > T_0. \tag{4.7}$$

Let $\breve{\mathbf{I}}^t$ be a solution of (4.1), (4.2) and (3.2) with initial datum $\mathbf{I}_0 \in \mathcal{F}_\mu(\mathbb{R}^+, \Omega)$. As a consequence of (4.1), (3.2), (4.3) and (4.6) we obtain

$$\frac{d}{dt}\Psi(t) \leq \frac{1}{4} \int_0^\infty \int_\Omega \frac{\xi(\tau)}{\mu'(\mathbf{x}, \tau)} \left| \frac{\partial}{\partial \tau} \breve{\mathbf{I}}^t(\mathbf{x}, \tau) \right|^2 d\mathbf{x} d\tau \leq \frac{1}{4} \int_0^t \int_\Omega \frac{\xi(\tau)}{\mu'(\mathbf{x}, \tau)} \left| \frac{\partial}{\partial \tau} \breve{\mathbf{I}}^t(\mathbf{x}, \tau) \right|^2 d\mathbf{x} d\tau \leq 0. \tag{4.8}$$

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[2]We recall that, as proved in Fabrizio and Golden (2002), $\mathcal{H}_\mu(\mathbb{R}^+, \Omega)$ is the space where the minimal free energy is defined, while $\mathcal{F}_\mu(\mathbb{R}^+, \Omega)$ is the domain of the free energy introduced in Fabrizio (2004). Therefore $\mathcal{F}_\mu(\mathbb{R}^+, \Omega) \subset \mathcal{H}_\mu(\mathbb{R}^+, \Omega)$.

Finally, if $t > T_0$, the properties of ξ and the inequality (4.7) yield

$$\frac{d}{dt}\Psi(t) \le \frac{\xi(t)}{4}\int_0^t\int_\Omega \frac{1}{\mu'(\mathbf{x},\tau)}\left|\frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\mathbf{x},\tau)\right|^2 d\mathbf{x}d\tau \le \frac{\xi(t)}{4\alpha_{T_0}}\int_0^\infty\int_\Omega \frac{1}{\mu'(\mathbf{x},\tau)}\left|\frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\mathbf{x},\tau)\right|^2 d\mathbf{x}d\tau = -\frac{\xi(t)}{\alpha_{T_0}}\Psi(t) \tag{4.9}$$

and the integration of (4.9) gives

$$\Psi(t) \le \Psi(T_0)\exp\left[-\frac{1}{\alpha_{T_0}}\int_{T_0}^t \xi(s)ds\right] \le \Psi(\mathbf{I}_0)\exp\left[-\frac{1}{\alpha_{T_0}}\int_{T_0}^t \xi(s)ds\right], \quad t > T_0. \tag{4.10}$$

We conclude this section by stating the following theorem

Theorem 4.1. *Let* $\mathbf{v}$ *be a virtual work solution of problem* $(3.1) - (3.2)$ *with a vanishing external source and* $\breve{\mathbf{I}}^0 \in \mathcal{F}_\mu(\mathbb{R}^+,\Omega)$. *If* μ *satisfies* (4.3), *then, for* $T_0 > 0$, *there exist two positive constants* α_{T_0} *and* β_{T_0} *such that*

$$\Psi(t) \le \beta_{T_0}\Psi(0)\exp\left[-\frac{1}{\alpha_{T_0}}\int_0^t \xi(s)ds\right], \quad t > T_0. \tag{4.11}$$

Corollary 4.1. *Under the hypotheses of* Theorem 4.1, *the energy functional* (4.5) *exponentially (polinomially) decays if the memory kernel* μ *exponentially (polinomially) decays in time.*

Proof. It is easy to show that if ξ is constant in time, then (4.3) assures the exponential decay of μ, while (4.11) yields the exponential decay of the energy.
On the other hand, if $\xi(t) = c(1+t)^{-1}$, then $-\mu'(t) = O((1+t)^{-c})$ and from (4.11) we obtain

$$\Psi(t) \le \beta_{T_0}\Psi(0)(1+t)^{-c/\alpha_{T_0}}, \quad t > T_0.$$

$\square$

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